

# Coefficient Inequalities and Convolution Conditions

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**Abstract.** Some interesting sufficient conditions involving coefficient inequalities for functions belonging to the classes  $\mathcal{S}^*(A, B)$ ,  $\mathcal{K}(A, B)$ ,  $\mathcal{S}_\lambda^*(A, B)$  and  $\mathcal{K}_\lambda(A, B)$  are derived. Several known convolution conditions, coefficient inequalities are special cases and consequences of these coefficient inequalities.

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## 1. INTRODUCTION

Let  $\mathcal{N}$  denote the class of normalized analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

defined in the unit disc  $\mathcal{U} = \{z : |z| < 1\}$ . For any two real numbers  $A$  and  $B$ ,  $-1 \leq B < A \leq 1$ , the class  $\mathcal{P}(A, B)$  [1] consists of functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

which are analytic in the unit disc  $\mathcal{U}$  such that

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \quad z \in \mathcal{U}$$

where  $\omega(z)$  is analytic in  $\mathcal{U}$  satisfying the conditions  $\omega(0) = 0$ ,  $|\omega(z)| < 1$ ,  $z \in \mathcal{U}$ . The classes  $\mathcal{S}^*(A, B)$ ,  $\mathcal{K}(A, B)$ ,  $\mathcal{S}_\lambda^*(A, B)$  and  $\mathcal{K}_\lambda(A, B)$  are defined as follows:

$$\mathcal{S}^*(A, B) := \left\{ f(z) \in \mathcal{N} : \frac{zf'(z)}{f(z)} \in \mathcal{P}(A, B) \right\}$$

$$\begin{aligned}\mathcal{K}(A, B) &:= \left\{ f(z) \in \mathcal{N} : 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}(A, B) \right\} \\ \mathcal{S}_\lambda^*(A, B) &:= \left\{ f(z) \in \mathcal{N} : \frac{e^{i\lambda} \left( \frac{zf'}{f} \right) - i \sin \lambda}{\cos \lambda} \in \mathcal{P}(A, B) \right\} \\ \mathcal{K}_\lambda(A, B) &:= \left\{ f(z) \in \mathcal{N} : \frac{e^{i\lambda} \left( \frac{zf'}{f} \right)' - i \sin \lambda}{\cos \lambda} \in \mathcal{P}(A, B) \right\}\end{aligned}$$

For  $A = 1$  and  $B = -1$  we get the well known classes  $\mathcal{S}^*$ ,  $\mathcal{K}$ ,  $\mathcal{S}_\lambda^*$  and  $\mathcal{K}_\lambda$ .

## 2. PRELIMINARY LEMMAS

In this section we prove the following necessary and sufficient conditions.

**Lemma 2.1.** *A function  $p(z) \in \mathcal{P}(A, B)$  if and only if*

$$p(z) \neq \frac{1 + A\zeta}{1 + B\zeta} \quad (z \in \mathcal{U}, \zeta \in \mathbb{C}, |\zeta| = 1).$$

*Proof.* The proof is quite obvious. For, consider the linear transformation

$$\omega = \frac{1 + Az}{1 + Bz}$$

which maps the unit circle  $\delta\mathcal{U}$  onto the imaginary axis  $\Re\{\omega\} = 0$ . Indeed, for all  $\zeta$  such that  $|\zeta| = 1$  ( $\zeta \in \mathbb{C}$ ), set

$$\omega = \frac{1 + A\zeta}{1 + B\zeta} \quad (\zeta \in \mathbb{C}, |\zeta| = 1)$$

so that

$$|\zeta| = \left| \frac{\omega - 1}{A - \omega B} \right| = 1$$

which shows that  $\Re\{\omega\} = \Re\left\{ \frac{1 + Az}{1 + Bz} \right\} = 0$  ( $\zeta \in \mathbb{C}, |\zeta| = 1$ ). Since  $p(0) = 1$ , for  $p(z) \in \mathcal{P}(A, B)$  we know that

$$p(z) \neq \frac{1 + A\zeta}{1 + B\zeta} \quad (z \in \mathcal{U}, \zeta \in \mathbb{C}, |\zeta| = 1)$$

which completes the proof.  $\square$

**Lemma 2.2.** *A function  $f(z) \in \mathcal{N}$  is in the class  $\mathcal{S}^*(A, B)$  if and only if*

$$(2.1) \quad 1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0$$

where

$$A_n = \frac{(n-1) + (nB - A)\zeta}{\zeta(B - A)} a_n.$$

*Proof.* A function  $f(z) \in \mathcal{N}$  is in the class  $\mathcal{S}^*(A, B)$  if and only if

$$\frac{zf'(z)}{f(z)} \neq \frac{1+A\zeta}{1+B\zeta}$$

That is,

$$(1+B\zeta)(zf'(z)) - (1+A\zeta)f(z) \neq 0$$

which implies

$$(B-A)\zeta z + \sum_{n=2}^{\infty} [n(1+B\zeta) - (1+A\zeta)] a_n z^n \neq 0.$$

This simplifies into

$$(2.2) \quad (B-A)\zeta z \left[ 1 + \sum_{n=2}^{\infty} \frac{n(1+B\zeta) - (1+A\zeta)}{\zeta(B-A)} a_n z^{n-1} \right] \neq 0.$$

Dividing both sides of (2.2) by  $(B-A)\zeta z$  ( $z \neq 0$ ) we obtain

$$1 + \sum_{n=2}^{\infty} \frac{(n-1) + (nB-A)\zeta}{\zeta(B-A)} a_n z^{n-1} \neq 0 \quad (z \in \mathcal{U}, \zeta \in \mathbb{C}, |\zeta| = 1)$$

which completes the proof.  $\square$

**Remark 2.3.** It follows from the normalization conditions  $a_0 = 0$  and  $a_1 = 1$  that

$$A_0 = -\frac{(1+A\zeta)}{B-A} a_0 = 0, \quad A_1 = \frac{(1+B\zeta) - (1+A\zeta)}{(B-A)\zeta} a_1 = 1.$$

**Remark 2.4.** The assertion (2.1) of Lemma 2.2 is equivalent to

$$\frac{1}{z} \left( f(z) * \frac{\zeta z(B-A) + (1+A\zeta)z^2}{(1-z)^2} \right) \neq 0 \quad (|z| < R, |\zeta| = 1)$$

which was given earlier by Ganesan [2].

**Remark 2.5.** Further more, for  $A = 1$  and  $B = -1$  as a special case we get convolution conditions characterizing starlike functions as in [3] with a suitable modification.

### 3. MAIN RESULTS

In this section, we determine certain conditions for functions in the class  $\mathcal{N}$ .

**Theorem 3.1.** If  $f(z) \in \mathcal{N}$  satisfies the following condition

$$(3.1) \quad \left| \sum_{k=1}^{\infty} \left[ \sum_{j=1}^n \left( \sum_{j=1}^k (-1)^{k-j} (j-1) \binom{\beta}{k-j} a_j \right) \binom{\gamma}{n-k} \right] + \left| \sum_{k=1}^n \left( \sum_{j=1}^k (-1)^{k-j} (jB-A) \binom{\beta}{k-j} a_j \right) \binom{\gamma}{n-k} \right| \right] \right| \leq B-A$$

with  $\beta, \gamma \in \mathbb{R}$  and  $-1 \leq B < A \leq 1$  then  $f(z) \in \mathcal{S}^*(A, B)$ .

*Proof.* We note that  $(1-z)^\beta \neq 0$  and  $(1+z)^\gamma \neq 0$  for  $\beta, \gamma \in \mathbb{R}$  and  $z \in \mathcal{U}$ . If the inequality

$$(3.2) \quad \left(1 + \sum_{n=2}^{\infty} A_n z^{n-1}\right) (1-z)^\beta (1+z)^\gamma \neq 0 \quad (z \in \mathcal{U}, \beta, \gamma \in \mathbb{R})$$

holds true, then we have,

$$1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0,$$

which is the relation (2.1) of Lemma 2.2. Equation (2.1) is equivalent to

$$(3.3) \quad \left(1 + \sum_{n=2}^{\infty} A_n z^{n-1}\right) \left(\sum_{n=0}^{\infty} (-1)^n b_n z^n\right) \left(\sum_{n=0}^{\infty} c_n z^n\right) \neq 0$$

where,  $b_n = \binom{\beta}{n}$  and  $c_n = \binom{\gamma}{n}$ .

Considering the Cauchy product of the first two factors, expression (3.3) can be rewritten as

$$(3.4) \quad \left(1 + \sum_{n=2}^{\infty} B_n z^{n-1}\right) \left(\sum_{n=0}^{\infty} c_n z^n\right) \neq 0$$

where

$$B_n = \sum_{j=1}^n (-1)^{n-j} A_j b_{n-j}.$$

Further, by applying the Cauchy product again in (3.4) we find that

$$1 + \sum_{n=2}^{\infty} \left(\sum_{k=1}^n B_k c_{n-k}\right) z^{n-1} \neq 0 \quad (z \in \mathcal{U}).$$

Equivalently, we have

$$1 + \sum_{n=2}^{\infty} \left[ \left(\sum_{k=1}^n (-1)^{k-j} A_j b_{k-j}\right) c_{n-k} \right] z^{n-1} \neq 0 \quad (z \in \mathcal{U}).$$

If  $f(z) \in \mathcal{N}$  satisfies the following inequality

$$\sum_{n=2}^{\infty} \left| \sum_{k=1}^n \left(\sum_{j=1}^k (-1)^{k-j} A_j b_{k-j}\right) c_{n-k} z^{n-1} \right| \neq 0 \quad (z \in \mathcal{U}).$$

That is if

$$\begin{aligned} & \frac{1}{\zeta(B-A)} \sum_{n=2}^{\infty} \left| \sum_{k=1}^n \left(\sum_{j=1}^k (-1)^{k-j} [(j-1) + (jB-A)\zeta] a_j b_{k-j}\right) c_{n-k} \right| \\ & \leq \frac{1}{(B-A)} \sum_{n=2}^{\infty} \left( \left| \sum_{k=1}^n \left[ \sum_{j=1}^k (-1)^{k-j} (j-1) a_j b_{k-j} \right] c_{n-k} \right| + \right. \end{aligned}$$

$$\left| \sum_{k=1}^n \left[ \sum_{j=1}^k (-1)^{k-j} (jB - A) a_j b_{k-j} \right] c_{n-k} \right| \leq 1$$

for  $-1 \leq B < A \leq 1$ ,  $\zeta \in \mathbb{C}$ ,  $|\zeta| = 1$  then,  $f(z) \in \mathcal{S}^*(A, B)$  which establishes the result.  $\square$

**Corollary 3.2.** For  $A = 1 - 2\alpha$  and  $B = -1$  we get coefficient conditions for functions in the class  $\mathcal{S}^*(\alpha)$  [2] with suitable modifications.

Moreover, for  $\beta = \gamma = 0$ ,  $A = 1$ ,  $B = -1$  in Theorem 3.1 we obtain the following result.

**Corollary 3.3.** If  $f(z) \in \mathcal{N}$  satisfies the following coefficient inequality

$$(3.5) \quad \sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq 1 - \alpha \quad (0 \leq \alpha < 1)$$

then  $f(z) \in \mathcal{S}^*(\alpha)$ .

In particular, by putting  $\alpha = 0$  in (3.5) we get the following well - known coefficient condition for the class of starlike functions in  $\mathcal{U}$ .

**Corollary 3.4.** If  $f(z) \in \mathcal{N}$  satisfies

$$(3.6) \quad \sum_{n=2}^{\infty} n |a_n| \leq 1$$

then  $f(z) \in \mathcal{S}^*$ .

The following theorem gives the coefficient condition for functions  $f(z)$  to be in the class  $\mathcal{K}(A, B)$ .

**Theorem 3.5.** If  $f(z) \in \mathcal{N}$  satisfies the condition

$$(3.7) \quad \sum_{n=2}^{\infty} \left[ \left| \sum_{k=1}^n \left( \sum_{j=1}^k (-1)^{k-j} j(j-1) \binom{\beta}{k-j} a_j \right) \binom{\gamma}{n-k} \right| + \right. \\ \left. \left| \sum_{k=1}^n \left( \sum_{j=1}^k (-1)^{k-j} j(jB - A) \binom{\beta}{k-j} a_j \right) \binom{\gamma}{n-k} \right| \right] \leq B - A$$

for  $-1 \leq B < A \leq 1$ ,  $\beta, \gamma \in \mathbb{R}$ , then  $f(z) \in \mathcal{K}(A, B)$ .

*Proof.* Since  $f(z)$  is in the class  $\mathcal{K}(A, B)$  if and only if  $zf'(z)$  belongs to the class  $\mathcal{S}^*(A, B)$ , replacing  $a_j$  in the statement of the Theorem 3.1 by  $ja_j$  we get the required result.  $\square$

**Corollary 3.6.** For  $A = 1 - 2\alpha$  and  $B = -1$  we get coefficient conditions for functions in the class  $\mathcal{K}(\alpha)$  [2] with suitable modifications.

Moreover, for  $\beta = \gamma = 0$ ,  $A = 1$ ,  $B = -1$  in Theorem 3.5 we obtain the following result.

**Corollary 3.7.** *If  $f(z) \in \mathcal{N}$  satisfies the following coefficient inequality*

$$(3.8) \quad \sum_{n=2}^{\infty} n(n-\alpha) |a_n| \leq 1 - \alpha \quad (0 \leq \alpha < 1)$$

*then  $f(z) \in \mathcal{K}(\alpha)$ .*

In particular, by putting  $\alpha = 0$  in (3.5) we get the following well - known coefficient condition for the class of convex functions in  $\mathcal{U}$ .

**Corollary 3.8.** *If  $f(z) \in \mathcal{N}$  satisfies*

$$(3.9) \quad \sum_{n=2}^{\infty} n^2 |a_n| \leq 1$$

*then  $f(z) \in \mathcal{K}$ .*

#### 4. COEFFICIENT CONDITIONS FOR FUNCTIONS IN THE CLASSES

$\mathcal{S}_{\lambda}^*(A, B)$  AND  $\mathcal{K}_{\lambda}(A, B)$ .

In this section, we obtain coefficient conditions for functions belonging to the classes  $\mathcal{S}_{\lambda}^*(A, B)$  and  $\mathcal{K}_{\lambda}(A, B)$ .

**Lemma 4.1.** *A function  $f(z) \in \mathcal{N}$  is in the class  $\mathcal{S}_{\lambda}^*(A, B)$  if and only if*

$$(4.1) \quad 1 + \sum_{n=2}^{\infty} d_n z^{n-1} \neq 0$$

*where*

$$d_n = \frac{(n-1) + (nB - \gamma)\zeta}{\zeta(B-A)} a_n \text{ and } \gamma = (A \cos \lambda + iB \sin \lambda)e^{-i\lambda}.$$

*Proof.* A function  $f(z) \in \mathcal{N}$  is in the class  $\mathcal{S}_{\lambda}^*(A, B)$  if and only if

$$e^{i\lambda} \frac{\frac{zf'}{f} - i \sin \lambda}{\cos \lambda} \neq \frac{1 + A\zeta}{1 + B\zeta}.$$

This simplifies into

$$(1 + B\zeta)(zf'(z)) - (1 + \gamma\zeta)f(z) \neq 0$$

where  $\gamma = (A \cos \lambda + iB \sin \lambda)e^{-i\lambda}$ . The rest of the proof follows as in Lemma 2.2.  $\square$

**Theorem 4.2.** If  $f(z) \in \mathcal{N}$  satisfies the following condition:

$$\sum_{n=2}^{\infty} \left[ \left| \sum_{k=1}^n \left( \sum_{j=1}^k (-1)^{k-j} (jB - \gamma) \binom{\beta}{k-j} a_j \right) \binom{\gamma}{n-k} \right| + \left| \sum_{k=1}^n \left( \sum_{j=1}^k (-1)^{k-j} (j-1) \binom{\beta}{k-j} a_j \right) \binom{\gamma}{n-k} \right| \right] \leq B - A$$

for  $-1 \leq B < A \leq 1$  then,  $f(z) \in \mathcal{S}_{\lambda}^*(A, B)$  where  $\gamma = (A \cos \lambda + iB \sin \lambda)e^{-i\lambda}$ .

*Proof.* Applying the same methods as in Theorem 3.1 we get the result.  $\square$

**Theorem 4.3.** If  $f(z) \in \mathcal{N}$  satisfies the following condition:

$$\sum_{n=2}^{\infty} \left[ \left| \sum_{k=1}^n \left( \sum_{j=1}^k (-1)^{k-j} j(jB - \gamma) \binom{\beta}{k-j} a_j \right) \binom{\gamma}{n-k} \right| + \left| \sum_{k=1}^n \left( \sum_{j=1}^k (-1)^{k-j} j(j-1) \binom{\beta}{k-j} a_j \right) \binom{\gamma}{n-k} \right| \right] \leq B - A$$

for  $-1 \leq B < A \leq 1$  then,  $f(z) \in \mathcal{K}_{\lambda}(A, B)$  where  $\gamma = (A \cos \lambda + iB \sin \lambda)e^{-i\lambda}$ .

*Proof.* Since  $f(z)$  is in the class  $\mathcal{K}_{\lambda}(A, B)$  if and only if  $zf'(z)$  belongs to the class  $\mathcal{S}_{\lambda}^*(A, B)$ , replacing  $a_j$  in the statement of the Theorem 4.2 by  $ja_j$  we get the required result.  $\square$

**Corollary 4.4.** For parametric values  $A = 1 - 2\alpha$  and  $B = -1$  we get coefficient conditions for functions in the class  $\mathcal{S}_{\lambda}^*(\alpha)$  and  $\mathcal{K}_{\lambda}(\alpha)$  [2] with suitable modifications.

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